Local optima and classification

Find the local optima and classify them for the following function: $f(x, y) = 8x^3 + 2xy - 3x^2 + y^2 + 1$.

Solution

First we take the derivatives:

$$f_1'(x,y) = 24x^2 + 2y - 6x,$$
$$f_2'(x,y) = 2x + 2y.$$

To determine the stationary points of the function, the first-order derivatives need to be set to zero. The second derivative equation yields $y = -x$. When we replace this in the first derivative equation, we obtain $24x^2 - 8x = 8x(3x-1) = 0$. Solving this results in two solutions, $x = 0$ and $x = \frac{1}{3}$, leading to two corresponding stationary points.

$$(x^*, y^*) = (0,0),$$
$$(x^{**}, y^{**}) = \left(\frac{1}{3}, -\frac{1}{3}\right).$$

To classify those points we take the second derivatives:

$$f_{11}''(x,y) = 48x - 6,$$
$$f_{22}''(x,y) = 2,$$
$$f_{12}''(x,y) = f_{21}''(x,y) = 2,$$

so Hessian is

$$H = \begin{pmatrix} f_{11}''(x,y) & f_{12}''(x,y) \\ f_{21}''(x,y) & f_{22}''(x,y \end{pmatrix} = \begin{pmatrix} 48x-6 & 2 \\ 2 & 2 \end{pmatrix}$$

Look at each stationary point in turn.
For $(x^*, y^*) = (0,0)$:

$$f_{11}''(0,0) = -6 < 0,$$
$$f_{11}''(0,0)f_{22}''(0,0) - (f_{12}''(0,0))^2 = -16 < 0.$$

So $(x^*, y^*) = (0,0)$ is neither a local maximizer nor a local minimizer (i.e. it is a saddle point).
For $(x^{**}, y^{**}) = \left(\frac{1}{3}, -\frac{1}{3}\right)$:

$$f_{11}''\left(\frac{1}{3}, -\frac{1}{3}\right) = 10 > 0,$$
$$f_{11}''\left(\frac{1}{3}, -\frac{1}{3}\right) f_{22}''\left(\frac{1}{3}, -\frac{1}{3}\right) - (f_{12}''\left(\frac{1}{3}, -\frac{1}{3}\right))^2 = 96/3 - 16 = 16 > 0.$$

So $(x^{**}, y^{**}) = \left(\frac{1}{3}, -\frac{1}{3}\right)$ is a local minimizer, with $f\left(\frac{1}{3}, -\frac{1}{3}\right) = 23/27$.